# Interior Penalty Discontinuous Galerkin Method for Magnetostatic Field Problems in Two Dimensions

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**An interior penalty discontinuous Galerkin method for solving two-dimensional magnetostatic field problems using the magnetic vector potential $\boldsymbol{A}$ is presented. The use of the $\boldsymbol{A}$-formulation in two dimensions results in a second-order elliptic boundary value problem. Due to the method, the scheme is symmetric and the resulting mass-matrix is block-diagonal, whereas each block belongs to one element of the triangulation. The applicability of the proposed method is demonstrated by solving a typical magnetostatic field problem and the numerical results are compared with the solution obtained from the standard finite element analysis.**

***Index Terms*—discontinuous Galerkin method, elliptic problem, interior penalty method, magnetostatics**

## I. Motivation

DISCONTINUOUS Galerkin (DG) methods offer a new perspective to simulate complex and large electromagnetic field problems. Because of their local formulation, their ability to handle non-matching meshes (e.g. hanging nodes) and the use of discontinuous shape functions with varying polynomial degree, DG methods are predestined for calculating electromagnetic phenomena. If the magnetic vector potential $\boldsymbol{A}$ is used to describe the magnetic field quantities in a two-dimensional region, the resulting problem is elliptic and of second order. There exist several DG formulations for second-order elliptic problems, see [1]. The method used here is derived from the interior penalty methods described in [2]. One motivation to use DG methods is the fact that all elements of the triangulation are decoupled and the resulting mass-matrix is block-diagonal, which is easier to invert than the common finite element mass-matrix. Information exchange between two neighbouring elements is guaranteed by the so-called numerical flux. Choosing the numerical flux appropriately leads to a stable numerical scheme and the decoupled elements make the method suitable for local refinement strategies without looking for neighbouring elements. In order to show the applicability of the DG method, a C-shaped magnet with linear ferromagnetic material is computed.

## II. Formulation

### A. Boundary Value Problem (BVP) and $\boldsymbol{A}$-formulation

The differential equations for stationary fields can be derived from Maxwell's equations by assuming time-independence ($\partial/\partial t = 0$) of the field quantities. Due to this, the magnetostatic field is described by (e.g. [3]):

$$\begin{gathered} \nabla \times \boldsymbol{H} = \boldsymbol{s}_i \\ \nabla \cdot \boldsymbol{B} = 0 \end{gathered} \tag{1}$$

In (1), $\boldsymbol{B}$ is the magnetic flux density, $\boldsymbol{H} = \nu \boldsymbol{B}$ is the magnetic field strength, $\nu$ is the reluctivity and $\boldsymbol{s}_i$ is the imposed current density. Consider a two-dimensional region $\Omega$ in the $x$-$y$ plane with boundary $\partial\Omega = \Gamma_B \cup \Gamma_H$, divided into two parts in accordance with the two types of boundary conditions [4]. The imposed current density $\boldsymbol{s}_i = s(x,y)\boldsymbol{e}_z$ is oriented in $z$ direction, which yields for the magnetic vector potential:

$$\boldsymbol{A} = A(x,y)\boldsymbol{e}_z \tag{2}$$

Using $A(x,y) = A$ and include $\boldsymbol{B} = \nabla \times \boldsymbol{A} = (\partial A/\partial y)\boldsymbol{e}_x - (\partial A/\partial x)\boldsymbol{e}_y$ into (1), the two-dimensional magnetic field is described by the magnetic vector potential as follows:

$$-\nabla \cdot (\nu(|\nabla A|)\nabla A) = s, \quad \text{in } \Omega \tag{3}$$

$$A = g_D, \quad \text{on } \Gamma_B \tag{4}$$

$$\nu(|\nabla A|)\nabla A \cdot \boldsymbol{n} = g_N, \quad \text{on } \Gamma_H \tag{5}$$

The reluctivity $\nu$ describes the constitutive relation between $\boldsymbol{B}$ and $\boldsymbol{H}$. In general $\nu$ is inhomogeneous, anisotropic, and non-linear $-$ depending on the field quantities. If a homogeneous, isotropic, and non-linear behaviour is assumed the reluctivity is a function of $|\nabla A|$. If the reluctivity is considered to be constant, (3) becomes a linear problem. Furthermore the BVP is defined by the inhomogeneous Dirichlet- and Neumann-boundary conditions (4) and (5) (e.g. [4]).

### B. Symmetric Interior Penalty Galerkin (SIPG) Formulation

The region $\Omega$ is decomposed into $k = 1, ..., K$ elements, naming one single element as $D^k$. Using a DG method for spatial discretisation means to search for a solution that is continuous inside one element and discontinuous across the element interfaces. Consequently the usual way to perform the variational formulation could not been followed. Integration by parts of (3) has to be performed over each element $D^k$ and not on the whole domain $\Omega$. Multiplying (3) by a test function $v$, integrating over one element $D^k$ and summing over all $k$ elements yields:

$$\begin{aligned} &\sum_k \int_{D^k} \nu \nabla A \cdot \nabla v \; d\boldsymbol{r} \\ &\quad - \sum_k \int_{\partial D^k} \nu \nabla A \cdot v \cdot \boldsymbol{n} \; ds = \sum_k \int_{D^k} s v \; d\boldsymbol{r} \end{aligned} \tag{6}$$

In (6) the reluctivity is supposed to be homogeneous, isotropic, and linear for each material. Fig. 1 shows two neighbouring elements $D^+$ and $D^-$ sharing one side $e$ with the unit outward normal vectors $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$, respectively.

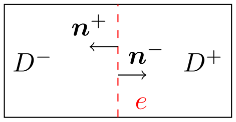


Fig. 1. Neighbouring elements $D^+$ and $D^-$ sharing one common edge $e$.

By defining $\epsilon^0$ the set of internal edges, $\epsilon^D$ the set of Dirichlet-boundary edges and $\epsilon^N$ the set of Neumann-boundary edges, the second term in (6) can be reformulated as an integral over internal and boundary edges. Note the average $\{\{\cdot\}\}$ and jump $[[\cdot]]$ of a related function at any point $p$ of $e \in \epsilon^0$ are defined as:

$$\{\{u\}\} := \left(u^- + u^+\right)/2, \;\; [[u]] := u^- \boldsymbol{n}^- + u^+ \boldsymbol{n}^+ \quad (7)$$

If $p$ is a point of $e \in \epsilon^D \cup \epsilon^N$ the function is single valued, $\{\{u\}\} := u^+$ and $[[u]] := u^+ \boldsymbol{n}^+$. Furthermore the exact solution $A$ in $\Omega$ is smooth on interior edges, $[[A]] = 0$, and satisfies (4) on boundary edges, $[[A - g_D]] = 0$. This fact leads to a term that could be added to the formulation without destroying consistency and making the problem symmetric:

$$-\int_{\epsilon^0} \{\{\nu \nabla v\}\} \llbracket A \rrbracket \; ds - \int_{\epsilon^D} \{\{\nu \nabla v\}\} \llbracket A - g_D \rrbracket \; ds \quad (8)$$

To penalize the discontinuity of the solution the penalty term

$$\int_{\epsilon^0} \sigma \llbracket A \rrbracket \llbracket v \rrbracket \;\; ds + \int_{\epsilon^D} \sigma \llbracket A - g_D \rrbracket \llbracket v \rrbracket \;\; ds \quad (9)$$

with the penalty parameter $\sigma$ is added to the form. The parameter $\sigma$ is a real non-negative number and calculated by the polynomial degree $N$ of the shape function, the $d$ and $(d-1)$ dimensional Hausdorff measure on each element $k$:

$$\sigma = N\left(N+1\right) \cdot |e^k|_{d-1}/|D^k|_d \quad (10)$$

If the corresponding edge is an interior one, the average $\sigma = (\sigma^+ + \sigma^-)/2$ is taken. The reformulation of (6) using (7) and adding (8) and (9) results in the SIPG formulation of (3):

$$\sum_k \int_{D^k} \nu \nabla A \nabla v \, d\boldsymbol{r} \quad (11)$$
$$-\sum_k \int_{\epsilon^0 \cup \epsilon^D} \left(\{\{\nu \nabla A\}\} \llbracket v \rrbracket + \{\{\nu \nabla v\}\} \llbracket A \rrbracket - \sigma \llbracket A \rrbracket \llbracket v \rrbracket\right) ds$$
$$= \sum_k \left( \int_{D^k} s v \, d\boldsymbol{r} + \int_{\epsilon^N} g_N v \, ds + \int_{\epsilon^D} \left(\nu \nabla v g_D \boldsymbol{n} + \sigma g_D v\right) ds \right)$$

The second term on the right hand side in (11) represents the natural boundary condition (5).

## III. Numerical results

The SIPG is applied to a model problem, which consists of the calculation of the magnetic field in a two dimensional C-shaped ferromagnetic material (blue) with a coil (red) surrounding one part of the magnetic yoke, as shown in Fig. 2.

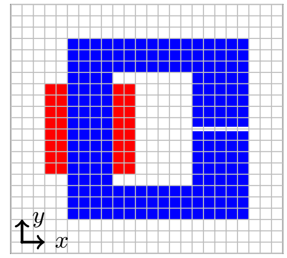


Fig. 2. C-shaped magnet (blue) and the surrounding coil (red).

The implementation is realised by using the open source finite element software library deal.II, see [5]. Homogeneous Dirichlet-boundary conditions $(g_D = 0)$ are assumed on the whole domain and the reluctivity jumps from $\nu_0 = 1/\mu_0$ in air to $\nu = \nu_0/2000$ in the ferromagnetic part. The impressed current density $|\boldsymbol{s}_i|$ is zero outside the coil and $|\boldsymbol{s}_i| = 263 \cdot 10^3$ A/m$^2$ inside. Using a quadrilateral mesh and Lagrange polynomials of second order as shape functions (see [5]) yields the solution $A$ from the SIPG as shown in Fig. 3. Computing the solution $A$ with continuous Lagrange elements leads to a magnetic field strength $|\boldsymbol{H}| = 734$ A/m in the air gap of the magnet. The resulting field strength derived from $A$ calculated by the SIPG is $|\boldsymbol{H}| = 736$ A/m.

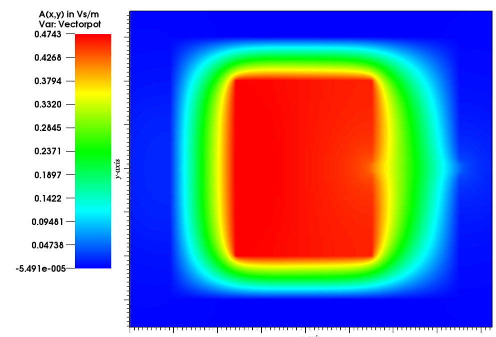


Fig. 3. Numerical result for the magnetic vector potential $A$ obtained from the SIPG.

## IV. Conclusion

The numerical results for linear ferromagnetic material indicate that the introduced method yields equal results concerning field strength and continuity of field quantities. A comprehensive review of the numerical results will be developed in the full paper. Furthermore the handling of non-linear ferromagnetic material within the SIPG will be treated in the extended version of the paper.

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